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THE SCRAMBLING INDEX OF PRIMITIVE DIGRAPHS CONSISTING OF A DIRECTED CYCLE AND A BI-DIRECTION PATH

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Abstract. *A strongly connected digraph is said tbe primitive provided there is a positive integer k such that for each pair of vertices u and v there are a walk from u to v and a walk from v to u of length k. The scrambling index of a primitive digraph D, denoted by $k(D)$, is the smallest positive integer k such that for each pair of vertices u and v there is a vertex w with the property that that there are a walk from u to w and a walk for v to w of length k. We discuss the scrambling index of the class of primitive digraph on n vertices consisting of a directed cycle of odd length s and a bi directed path of length $n - s$. For such primitive digraph D we show that $k(D) = 2s - 2$ whenever $s > n - s$ and $k(D) = n - 1$ whenever $s \leq n - s$.*

1. INTRODUCTION

Let D be a digraph. A directed walk of length m from a vertex u to a vertex v in D is a sequence of m arcs of the form $u = v_0 \to v_1 \to v_2 \to \cdots \to v_m = v$. For simplicity a directed walk of length m from a vertex u to a vertex v is denoted by $u \xrightarrow{m} v$. A $u \to v$ directed walk is open whenever $u \neq v$ and is closed whenever $u = v$. A $u \to v$ directed path is a $u \to v$ directed walk without repeated vertices except possibly $u = v$. A directed cycle is a closed directed path. The distance from a

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vertex u to a vertex v, denoted by $d(u,v)$, is the length of the shortest $u \to v$ path. A digraph D is called symmetric provided that $u \to v$ is an arc of D whenever $v \to u$ is an arc of D. By a bi-direction path or bi-path connecting vertices u and v we mean a walk connecting u and v of the form $u = v_0 \to v_1 \to v_2 \to \cdots \to v_m = v \to v_{m-1} \to v_{m-2} \to \cdots \to v_0 = u$.

A digraph D is strongly connected provided that for each pair of vertices u and v in D there are a $u \to v$ walk and a $v \to u$ walk. A strongly connected digraph is primitive provided there exists a positive integer k such that for each pair of vertices u and v in D there are a $u \xrightarrow{k} v$ walk and a $v \xrightarrow{k} u$ walk. The smallest of such positive integer k is called the exponent of D and is denoted by $\exp(D)$. Some result on the exponent of primitive digraph can be found on [1, 2]. It is a well known result, see [1], that a strongly connected digraph D is primitive if and only if the greatest common divisor of the lengths of all directed cycles in D is 1.

In 2009, Akelbek and Kirkland [3] introduced a new parameter of primitive digraphs called scrambling index. The scrambling index of a primitive digraph D, denoted by $k(D)$, is the smallest positive integer k such that for each pair of vertices u and v there exists a vertex w with the property that there are a $u \xrightarrow{k} w$ walk and a $v \xrightarrow{k} w$ walk in D. Results on scrambling index of primitive digraphs can be found on [4, 5, 6, 7, 8]. Liu and Huang [8] discussed the scrambling index of symmetric digraphs. In particular, they show that if D is a $(n, s, n-s)$-lollipop, that is a primitive symmetric digraph on n vertices consisting of bi-direction cycle of odd length s and bi-direction path of length $n-s$ as shown in Figure 1, then $k(D) = n - (s+1)/2$.

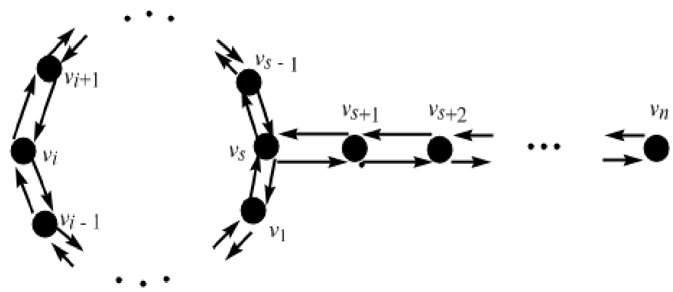

Figure 1: $(n, s, n-s)$-lollipop.

In this paper we discuss the scrambling index of an $(n, s, n-s)$-cycle bipath D_s^{n-s}, that is primitive digraphs on n vertices consisting of a directed cycle of odd length s and a bi-direction path of length $n-s$. We label the n

vertices of D_s^{n-s} as shown in Figure 2. In Section 2 we discuss a necessary background on scrambling index. In Section 3, we present our result.

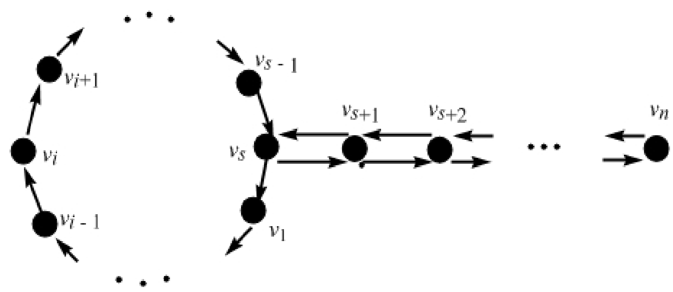

Figure 2: $(n, s, n-s)$-cycle bi-path.

2. NECESSARY BACKGROUND

Let D be a primitive digraph and u and v be two distinct vertices in D. The *local scrambling index* of u and v at the vertex w, denoted by $k_{u,v}(w)$, is defined to be the smallest positive integer k such that there are $u \xrightarrow{k} w$ walk and $v \xrightarrow{k} w$ walk. That is

$$k_{u,v}(w) = \min\{k : \text{there are } u \xrightarrow{k} w \text{ walk and } v \xrightarrow{k} w \text{ walk}\}. \qquad (1)$$

The *local scrambling index* of u and v in D, denoted by $k_{u,v}(D)$, is defined to be

$$k_{u,v}(D) = \min_{w \in V}\{k_{u,v}(w)\}. \qquad (2)$$

From the definition of $k(D)$ and $k_{u,v}(D)$ we have for any pair of distinct vertices u and v that $k(D) \geq k_{u,v}(D)$. Furthermore, since D is strongly connected for any positive integer $\ell \geq k_{u,v}(D)$ one can find a vertex w' with the property that there are $u \xrightarrow{\ell} w'$ walk and $v \xrightarrow{\ell} w'$. This implies

$$k(D) = \max_{u \neq v}\{k_{u,v}(D)\}. \qquad (3)$$

The following corollary presents primitivity of an $(n, s, n-s)$-cycle bi-path.

Corollary 2.1 *Let D_s^{n-s} be an $(n, s, n-s)$-cycle bi-path. If s is odd, then D_s^{n-s} is primitive.*

Proof. Notice that every bi-path $v_x \to v_{x+1} \to v_x$, for some $s \le x \le n-1$, is a directed cycle of length 2. Since s is odd, the greatest common divisor of lengths of cycles in D_s^{n-s} is 1. Hence D_s^{n-s} is primitive. ■

3. MAIN RESULT

Let D_s^{n-s} be an $(n, s, n-s)$-cycle bi-path. In this section we discuss an upper bound for $k(D_s^{n-s})$. For this purpose we define C_s to be the directed cycle of length s, $C_s : v_1 \to v_2 \to \cdots \to c_s \to v_1$ and P_{n-s} be the bi-path of length $2(n-s)$, $P_{n-s} : v_s \to v_s{+}1 \to \cdots \to v_{n-1} \to v_n \to v_{n-1} \to v_{n-1} \to \cdots \to v_{s+1} \to v_s$.

Theorem 3.1 *Let n be a positive integer with $n \ge 4$ and s be an odd positive integer with $s < n$. Then*

$$k(D_s^{n-s}) = \begin{cases} 2s-2 & \text{if } s > n-s, \\ n-1 & \text{if } s \le n-s. \end{cases}$$

Proof. Since s is odd, by Corollary 2.1 D_s^{n-s} is a primitive digraph. We split the proof into two cases, where $s > n-s$ and $s \le n-s$.

Case 1. $s > n-s$

We first show that $k(D_s^{n-s}) \ge 2s-2$. By (3), it suffices to show that there are vertices u and v in D_s^{n-s} such that $k_{u,v}(D_s^{n-s}) = 2s-2$. We show that

$$k_{v_1,v_2}(D_s^{n-s}) = \min_{w \in V}\{k_{v_1,v_2}(w)\} = 2s-2.$$

We consider four sub cases depending on the position of the vertex w.

Sub Case 1.1. *The vertex $w = v_1$.* The length of any $v_1 \to v_1$ walk is of the form $s\,x_1 + 2\,y_1$ for some integers $x_1 \ge 1$ and $y_1 \ge 0$. The length of any $v_2 \to v_1$ walk is of the form $(s-1) + s\,x_2 + 2\,y_2$ for some integers $x_2, y_2 \ge 0$. Notice also that there are $v_1 \to v_1$ walk and $v_2 \to v_1$ walk of the same length whenever $s\,x_1 + 2\,y_1 = (s-1) + s\,x_2 + 2\,y_2$. Since this condition occurs the first time when $x_1 = 1$ and $x_2 = 1$, we have $2y_1 - 2y_2 = s-1$. Therefore, $2y_1 \ge s-1$ and hence $s\,x_1 + 2\,y_1 \ge 2s-1$. Thus $k_{v_1,v_2}(v_1) \ge 2s-1$.

We next show that $k_{v_1,v_2}(v_1) \le 2s-1$. The walk that starts at v_1, moves to v_s along the $v_1 \to v_s$ path, moves $(s-1)/2$ times around the cycle $v_s \to v_{s+1} \to v_s$, and finally moves to v_1 using the $v_s \to v_1$ arc is a $v_1 \to v_1$ walk of length $2s-1$. Notice also that the walk that starts at v_2, moves one time around the cycle C_s and back at v_2, then finally moves to

v_1 along the $v_2 \to v_1$ path is a $v_2 \to v_1$ walk of length $2s-1$. Therefore, there are $v_1 \xrightarrow{2s-1} v_1$ walk and $v_2 \xrightarrow{2s-1} v_1$ walk in D_s^{n-s}. Hence by (1) we have $k_{v_1,v_2}(v_1) \leq 2s-1$. Hence we now conclude that

$$k_{v_1,v_2}(v_1) = 2s-1. \tag{4}$$

Sub Case 1.2. *The vertex* $w = v_j$, *for* $j = 2, 3, \ldots, s-1$. The length of any $v_1 \to v_j$ walk is of the form $j-1+s\,x_1+2\,y_1$ for some $x_1, y_1 \geq 0$ and the length of any $v_2 \to v_j$ walk is of the form $j-2+s\,x_2+2\,y_2$ for some $x_2, y_2 \geq 0$. Notice also that there are $v_1 \to v_j$ walk and $v_2 \to v_j$ walk of the same length whenever $(j-1)+s\,x_1+2\,y_1 = (j-2)+s\,x_2+2\,y_2$ for some $x_1, x_2 \geq 1$ and $y_1, y_2 \geq 0$. Furthermore, $(j-1)+s\,x_1+2\,y_1 = (j-2)+s\,x_2+2\,y_2$ occurs the first time when $x_1 = 1$ and $x_2 = 2$ or when $x_1 = 2$ and $x_2 = 1$. If $x_1 = 1$ and $x_2 = 2$, then $2y_1 - 2y_2 = s-1$. Therefore, $2y_1 \geq (s-1)$ and hence $(j-1)+s\,x_1+2\,y_1 \geq 2s-2+j$. If $x_1 = 2$ and $x_2 = 1$, then $2y_2 - 2y_1 = s+1$. Therefore, $2y_2 \geq s+1$ and hence $(j-2)+s\,x_2+2\,y_2 \geq 2s-1+j$. Thus we conclude that $k_{v_1,v_2}(v_j) \geq 2s-2+j$ for $j = 2, 3, \ldots, s-1$.

We now show that for $j = 2, 3, \ldots, s-1$, $k_{v_1,v_2}(v_j) \leq 2s-2+j$. The walk that starts at v_1 moves to v_j along the $v_1 \to v_j$ path, then moves to v_s along the $v_j \to v_s$ path, then moves $(s-1)/2$ times around the cycle $v_s \to v_{s+1} \to v_s$ and back at v_s, then finally moves to v_j along the $v_s \to v_j$ path, is a $v_1 \to v_j$ path of length $2s-2+j$. Moreover, the walk that starts at v_2, then moves to v_j along the $v_2 \to v_j$ path, and finally moves two time around the cycle C_s and back at v_j, is a $v_2 \to v_j$ walk of length $2s-2+j$. Therefore, for $j = 2, 3, \ldots, s-1$, there are $v_1 \xrightarrow{2s-2+j} v_j$ walk anda $v_2 \xrightarrow{2s-2+j} v_j$ walk and hence by (1) we have $k_{v_1,v_2}(v_j) \leq 2s-2+j$.

We now conclude that

$$k_{v_1,v_2}(v_j) = 2s-2+j \geq 2s \tag{5}$$

for each $j = 2, 3, \ldots, s-1$.

Sub Case 1.3. *The vertex* $w = v_s$. Notice that the length of any $v_1 \to v_s$ walk in D_s^{n-s} is of the form $(s-1)+s\,x_1+2\,y_1$ for some integers $x_1, y_1 \geq 0$, and the length of any $v_2 \to v_s$ walk in D_s^{n-s} is of the form $(s-2)+s\,x_2+2\,y_2$ for some integers $x_2, y_2 \geq 0$. Notice also that the expression $(s-1)+s\,x_1+2\,y_1 = (s-2)+s\,x_2+2\,y_2$ can occur the first time either when $x_1 = 0$ and $x_2 = 1$ or when $x_1 = 1$ and $x_2 = 0$. If $x_1 = 0$ and $x_2 = 1$, then $2y_1 - 2y_2 = s-1$. This implies $2y_1 \geq s-1$ and hence $(s-1)+s\,x_1+2\,y_1 \geq 2s-2$. If $x_1 = 1$ and $x_2 = 0$, then $2y_2 - 2y_1 = s+1$.

This implies $2y_2 \geq s+1$ and hence $(s-2)+s\,x_2+2\,y_2 \geq 2s-1$. Thus we conclude that $k_{v_1,v_2}(v_s) \geq 2s-2$.

We now show that $k_{v_1,v_2}(v_s) \leq 2s-2$. The walk that starts at v_1 moves to v_s along the $v_1 \rightarrow v_s$ path and finally moves $(s-1)/2$ times around the cycle $v_s \rightarrow v_{s+1} \rightarrow v_s$ is a $v_1 \rightarrow v_s$ walk of length $2s-2$. The walk that starts at v_2 moves to v_s along the $v_2 \rightarrow v_s$ path and finally moves one time aound the cycle C_s is a $v_2 \rightarrow v_s$ walk of length $2s-2$. Therefore, there are $v_1 \xrightarrow{2s-2} v_s$ walk and $v_2 \xrightarrow{2s-2} v_s$ walk in D_s^{n-s}. By (1) we have $k_{v_1,v_2}(v_2) \leq 2s-2$.

Therefore, we now conclude that

$$k_{v_1,v_2}(v_s) = 2s-2. \tag{6}$$

Sub Case 1.4. *The vertex $w = v_j$ for $j = s+1, s+2, \ldots, n$.* Notice that the length of any $v_1 \rightarrow v_j$ walk is of the form $(j-1)+s\,x_1+2\,y_1$ for some integers $x_1, y_1 \geq 0$ and the length of any $v_2 \rightarrow v_j$ walk is of the form $(j-2)+s\,x_2+2\,y_2$ for some integers $x_2, y_2 \geq 0$. Notice that $(j-1)+s\,x_1+2\,y_1 = (j-2)+s\,x_2+2\,y_2$ occurs the first time either when $x_1 = 0$ and $x_2 = 1$ or when $x_1 = 1$ and $x_2 = 0$. If $x_1 = 0$ and $x_2 = 1$, then $2y_1 - 2y_2 = s-1$. This implies $2y_1 \geq (s-1)$ and hence $(j-1)+s\,x_1+2\,y_1 \geq s-2+j$. If $x_1 = 1$ and $x_2 = 0$, then $2y_2 - 2y_1 = s+1$. This implies $2y_2 \geq s+1$ and hence $(j-2)+s\,x_2+2\,y_2 \geq s-1+j$. Thus we conclude that $k_{v_1,v_2}(v_j) \geq s-2+j$ for $j = s+1, s+2, \ldots, n$.

We now show that for $j = s+1, s+2, \ldots, n$, $k_{v_1,v_2}(v_j) \leq s-2+j$. The walk that starts at v_1, then moves to v_j along the $v_1 \rightarrow v_j$ path and finally moves $(s-1)/2$ times around the cycle $v_j \rightarrow v_{j-1} \rightarrow v_j$ is a $v_1 \rightarrow v_j$ walk of length $s-2+j$. The walk that starts at v_2, then moves one time around the cycle C_s and back at v_2 and finally moves to v_j along the $v_2 \rightarrow v_j$ path is a $v_2 \rightarrow v_j$ walk of length $s-2+j$. Therefore, for $j = s+1, s+2, \ldots n$, there are $v_1 \xrightarrow{s-2+j} v_j$ walk and $v_2 \xrightarrow{s-2+j} v_j$ walk in D_s^{n-s} and hence by (1) we have $k_{v_1,v_2}(v_j) \leq s-2+j$.

We now conclude that since $j > s$

$$k_{v_1,v_2}(v_j) = s-2+j > 2s-2 \tag{7}$$

for each $j = s+1, s+2, \ldots, n$.

From (4), (5), (6), (7), and (2) we now conclude that $k_{v_1,v_2}(D_s^{n-s}) = \min_{w \in V}\{k_{v_1,v_2}(w)\} = 2s-2$. Hence

$$k(D_s^{n-s}) \geq k_{v_1,v_2}(D_s^{n-s}) = 2s-2 \tag{8}$$

whenever $s > n-s$.

We now show that $k(D_s^{n-s}) \leq 2s-2$. We show that for each vertex $v_j, j = 1, 2, \ldots, n$, there is a $v_j \xrightarrow{2s-2} v_s$ walk in D_s^{n-s}.

Since v_s lies on a cycle of length 2, there is a $v_s \xrightarrow{2s-2} v_s$. We consider vertex $v_j \neq v_s$. Since $s > n-s$, for any vertex v_j the distance $d(v_j, v_s) \leq s-1$. If $d(v_j, v_s) \equiv 2s-2 \pmod 2$, then by using the cycle $v_s \rightarrow v_{s+1} \rightarrow v_s$ we can extend the $v_j \xrightarrow{d(v_j,v_s)} v_s$ path into a $v_j \xrightarrow{2s-2} v_s$ walk. If $d(v_j, v_s) \not\equiv \pmod 2$, then $d(v_j, v_s) \leq s-2$. Notice that the walk W_{v_j,v_s} that starts at v_j moves to v_s along the $v_j \xrightarrow{d(v_j,v_s)} v_s$ path and finally moves one time around the cycle C_s is a $v_j \rightarrow v_s$ walk of length $s + d(v_j, v_s) \equiv 2s-2 \pmod 2$. Since $s + d(v_j, v_s) \leq 2s-2$, we can extend the walk W_{v_j,v_s} into a $v_j \xrightarrow{2s-2} v_s$ walk. Therefore, for each $j = 1, 2, \ldots, n$, there is a $v_j \xrightarrow{2s-2} v_s$ walk. This implies

$$k(D_2^{n-s}) \leq 2s-2 \tag{9}$$

whenever $s > n-s$.

From (8) and (9), we now conclude that $k(D_s^{n-s}) = 2s-2$ whenever $s > n-s$.

Case 2. $s \leq n-s$
We first show that $k(D_2^{n-s}) \geq n-1$. By (3) it suffices to show that there are vertices u and v in D_s^{n-s} with the property that $k_{u,v}(D_s^{n-s}) = n-1$. We will show that

$$k_{v_{n-1},v_n}(D_s^{n-s}) = \min_{w \in V}\{k_{v_{n-1},v_n}(w)\} = n-1.$$

We consider four sub cases depending on the position of the vertex w.

Sub Case 2.1. *The vertex* $w = v_j$ *for some* $j = 1, 2, \ldots, s-1$. The length of any $v_{n-1} \rightarrow v_j$ walk is of the form $(n-1-s+j) + s\,x_1 + 2\,y_1$ for some $x_1, y_1 \geq 0$. The length of any $v_n \rightarrow v_j$ walk is of the form $(n-s+j) + s\,x_2 + 2\,y_2$ for some $x_2, y_2 \geq 0$. Notice that $(n-1-s+j) + s\,x_1 + 2\,y_1 = (n-s+j) + s\,x_2 + 2\,y_2$ occurs the first time either when $x_1 = 0$ and $x_2 = 1$ or when $x_1 = 1$ and $x_2 = 0$. If $x_1 = 0$ and $x_2 = 1$ we have $2y_1 - 2y_2 = s+1$ and hence $2y_1 \geq s+1$. This implies $(n-1-s+j) + s\,x_1 + 2\,y_1 \geq n+j$. If $x_1 = 1$ and $x_2 = 0$ we have $2y_2 - 2y_1 = s-1$ and hence $2y_2 \geq s-1$. This implies $(n-s+j) + s\,x_2 + 2\,y_2 \geq n+j-1$. Thus we conclude that $k_{v_{n-1},v_n}(v_j) \geq n-1+j$.

We now show, for $j = 1, 2, \ldots, s-1$, that $k_{v_{n-1},v_n}(v_j) \leq n-1+j$. The walk that starts at v_{n-1} moves to v_j along the $v_{n-1} \rightarrow v_j$ path and finally moves one time around the cycle C_s and back at v_j is a $v_{n-1} \to v_j$ of length $n-1+j$. The walk that starts at v_n moves $(s-1)/2$ times around the cycle $v_n \to v_{n-1} \to v_{n-1}$ and finally moves to v_j along the $v_n \to v_j$ path is a $v_n \to v_j$ walk of length $n-1+j$. Therefore, for $j = 1, 2, \ldots, s-1$, there are $v_{n-1} \overset{n-1+j}{\longrightarrow} v_j$ walk and $v_n \overset{n-1+j}{\longrightarrow} v_j$ walk in D_s^{n-s} and hence by (1) we have $k_{v_{n-1},v_n}(v_j) \leq n-1+j$.

We now conclude that

$$k_{v_{n-1},v_n}(v_j) = n-1+j \geq n \tag{10}$$

for each $j = 1, 2, \ldots, s-1$.

Sub Case 2.2. *The vertex* $w = v_s$. The length of any $v_n \to v_s$ walk is of the form $n - s + s\,x_1 + 2\,y_1$ for some integers $x_1, y_1 \geq 0$. The length of any $v_{n-1} \to v_s$ walk is of the form $(n-1-s) + s\,x_2 + 2\,y_2$ for some integers $x_2, y_2 \geq 0$. Notice that $n - s + s\,x_1 + 2\,y_1 = n - 1 - s + s\,x_2 + 2\,y_2$ occurs the first time either when $x_1 = 0$ and $x_2 = 1$ or when $x_1 = 1$ and $x_2 = 0$. If $x_1 = 0$ and $x_2 = 1$, then $2y_1 - 2y_2 = s - 1$ and hence $2y_1 \geq s - 1$. This implies $n-s+s\,x_1+2\,y_1 \geq n-1$. If $x_1 = 1$ and $x_2 = 0$, then $2y_2 - 2y_1 = s+1$ and hence $2y_2 \geq s+1$. This implies $n - s - 1 + s\,x_2 + 2\,y_2 \geq n$. Therefore, we now conclude that $k_{v_{n-1},v_n}(v_s) \geq n-1$.

We now show that $k_{v_{n-1},v_n}(v_s) \leq n-1$. The walk that starts at v_n, moves to v_s along the $v_n \overset{n-s}{\longrightarrow} v_s$ path and finally moves $(s-1)/2$ times around the cycle $v_s \to v_{s+1} \to v_s$ is a $v_n \to v_s$ walk of length $n-1$. The walk that starts at v_{n-1} moves to v_s along the $v_{n-1} \overset{n-s-1}{\longrightarrow} v_s$ path and finally moves one time around the cycle C_s is a $v_{n-1} \to v_s$ walk of length $n-1$. Therefore, there are $v_{n-1} \overset{n-1}{\longrightarrow} v_s$ walk and $v_n \overset{n-1}{\longrightarrow} v_s$ walk in D_s^{n-s} and hence by (1) we have $k_{v_{n-1},v_n}(v_s) \leq n-1$.

We now conclude that

$$k_{v_{n-1},v_n}(v_s) = n-1. \tag{11}$$

Sub Case 2.3. *The vertex* $w = v_j$ *for* $j = s+1, s+2, \ldots, n-1$. The length of any $v_n \to v_j$ walk is of the form $n - j + s\,x_1 + 2\,y_1$ for some integers $x_1, y_1 \geq 0$. The length of any $v_{n-1} \to v_j$ walk is of the form $n - j - 1 + s\,x_2 + 2\,y_2$ for some integers $x_2, y_2 \geq 0$. We note that $n - j + s\,x_1 + 2\,y_1 = n - j - 1 + s\,x_2 + 2\,y_2$ occurs the first time either when $x_1 = 0$ and $x_2 = 1$ or when $x_1 = 1$ and $x_2 = 0$. If $x_2 = 1$, then $y_2 \geq j - s$. This implies $n - j - 1 + s\,x_2 + 2\,y_2 \geq n - 1 + j - s$. If $x_1 = 1$, then $y_1 \geq j - s$. This implies $n - j + s\,x_1 + 2\,y_1 \geq n + j - s$. Therefore, for $j = s+1, s+2, \ldots, n-1$, we conclude that $k_{v_{n-1},v_n}(v_j) \geq n - 1 + (j - s)$.

We now show, for $j = s+1, s+2, \ldots, n-1$, that $k_{v_{n-1},v_n}(v_j) \leq n-1+j-s$. The walk that starts at v_n moves to v_j along the $v_n \stackrel{n-j}{\longrightarrow} v_j$ path and finally moves $j-(s+1)/2$ times around the cycle $v_j \rightarrow v_{j+1} \rightarrow v_j$ is a $v_n \rightarrow v_j$ walk of length $n-1+j-s$. The walk that starts at v_{n-1} moves to v_s along the $v_{n-1} \stackrel{n-1-s}{\longrightarrow} v_s$ path, then moves one time around the cycle C_s and finally moves to v_j along the $v_s \stackrel{j-s}{\longrightarrow} v_j$ path is a $v_{n-1} \rightarrow v_j$ walk of length $n-1+j-s$. Therefore, for $j = s+1, s+2, \ldots, n-1$, there are $v_{n-1} \stackrel{n-1+j-s}{\longrightarrow} v_j$ walk and $v_n \stackrel{n-1+j-s}{\longrightarrow} v_j$ walk in D_s^{n-s} and hence by (1) we have $k_{v_{n-1},v_n}(v_j) \leq n-1+j-s$.

We now conclude that

$$k_{v_{n-1},v_n}(v_j) = n-1+j-s \geq n \tag{12}$$

for each $j = s+1, s+2, \ldots, n-1$.

Sub Case 2.4. *The vertex $w = v_n$.* The length of any $v_n \rightarrow v_n$ walk is of the form $s\,x_1 + 2\,y_1$ for some integers $x_1, y_1 \geq 0$. The length of any $v_{n-1} \rightarrow v_n$ walk is of the form $1 + s\,x_2 + 2\,y_2$ for some integers $x_2, y_2 \geq 0$. We note that $s\,x_1 + 2\,y_1 = 1 + s\,x_2 + 2\,y_2$ occurs the first time either when $x_1 = 0$ and $x_2 = 1$ or when $x_1 = 1$ and $x_2 = 0$. If $x_1 = 1$, then $y_1 \geq n-s$. This implies $s\,x_1 + 2\,y_1 \geq 2n-s$. If $x_2 = 1$, then $y_2 \geq n-1-s$. This implies $1 + s\,x_2 + 2\,y_2 \geq 2n-s-1$. Therefore, we conclude that $k_{v_{n-1},v_n}(v_n) \geq 2n-s-1$.

We now show that $k_{v_{n-1},v_n}(v_n) \leq 2n-s-1$. The walk that starts at v_n moves $n-(s+1)/2$ times around the cycle $v_n \rightarrow v_{n-1} \rightarrow v_n$ is a $v_n \rightarrow v_n$ walk of length $2n-s-1$. The walk that starts at v_{n-1} moves to v_s along the $v_{n-1} \stackrel{n-1-s}{\longrightarrow} v_s$ path, then moves one time around the cycle C_s and back at v_s and finally moves to v_n along the $v_s \stackrel{n-s}{\longrightarrow} v_n$ is a $v_{n-1} \rightarrow v_n$ walk of length $2n-s-1$. Therefore, there are $v_{n-1} \stackrel{2n-s-1}{\longrightarrow} v_n$ walk and $v_n \stackrel{2n-s-1}{\longrightarrow} v_n$ walk in D_s^{n-s} and hence by (1) we have $k_{v_{n-1},v_n}(v_n) \leq 2n-s-1$.

We now conclude that

$$k_{v_{n-1},v_n}(v_n) = 2n-s-1 \geq n-1+s \geq n \tag{13}$$

since $n-s \geq s \geq 1$.

From (10), (11), (12), (13), and (2) we conclude that $k_{v_1,v_n}(D_s^{n-s}) = n-1$. This implies

$$k(D_s^{n-s}) \geq k_{v_{n-1},v_n}(D_s^{n-s}) = n-1 \tag{14}$$

whenever $n-s \geq s$.

We now show that $k(D_s^{n-s}) \le n-1$ whenever $n-s \ge s$. We show that for each vertex $v_j, j = 1, 2, \ldots, n$ there is a $v_j \stackrel{n-1}{\longrightarrow} v_s$ walk in D_s^{n-s}. Suppose $d(v_j, v_s) \equiv n-1 \pmod 2$. Since v_s lies on a cycle of length 2, then the $v_j \to v_s$ path of length $d(v_j, v_s)$ can be extended to a $v_j \stackrel{n-1}{\longrightarrow} v_s$ walk. So we assume that $d(v_j, v_s) \not\equiv n-1 \pmod 2$. Notice that the walk that starts at v_j moves to v_s along the $v_j \to v_s$ path and finally moves one time around the cycle C_s is a $v_j \to v_s$ walk of length $s + d(v_j, v_s) \equiv n-1 \pmod 2$. If v_j lies on the cycle C_s, then $s + d(v_j, v_s) \le 2s - 1$. Since $s \le n-s$, we now have $s + d(v_j, v_s) \le 2s - 1 \le n - 1$. If v_j does not lie on C_s that is v_j lies on P_{n-s}, then $d(v_j, v_s) \le n - 1 - s < n - 1$. Since v_s lies on a cycle of length 2 and $s + d(v_j, v_s) \equiv n-1 \pmod 2$, the $v_j \to v_s$ walk of length $s + d(v_j, v_s)$ can be extended to a $v_j \stackrel{n-1}{\longrightarrow} v_s$ walk. Therefore, for each $j = 1, 2, \ldots, n$, there is a $v_j \stackrel{n-1}{\longrightarrow} v_s$ walk in D_s^{n-s}. Hence we conclude that

$$k(D_s^{n-s}) \le n-1 \tag{15}$$

whenever $n - s \ge s$.

From (14) and (15) we conclude that $k(D_s^{n-s}) = n-1$ whenever $n-s \ge s$. ■

As a consequence of Theorem 3.1 we have a general upper bound for scrambling index of D_s^{n-s}.

Corollary 3.2 *Let $n \ge 4$ be an integer and let $s < n$ be an odd positive integer. Then*

$$k(D_s^{n-s}) \le \begin{cases} 2n-6, & \text{if } n \text{ is odd} \\ 2n-4, & \text{if } n \text{ is even.} \end{cases}$$

Proof. By Theorem 3.1 $k(D_s^{n-2})$ will be large whenever $s > n-s$. Notice that if n is even, then $s \le n-2$. This implies $k(D_s^{n-s}) = 2s - 2 \le 2n - 6$. If n is even, then $s \le n-1$. This implies $k(D_s^{n-s}) = 2s - 2 \le 2n - 4$. ■

We note that if n is odd, by Theorem 3.1, $k(D_s^{n-s}) = 2n-6$ is achieved by the digraph D_{n-2}^2. If n is even, $k(D_s^{n-s}) = 2n-4$ is achieved by the digraph D_{n-1}^1.

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