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Bounds for scrambling index of primitive graphs

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Abstract. A connected graph is primitive provided there is a positive integer m such that for each pair of vertices u and v there is a walk of length m connecting u and v. The scrambling index of a primitive graph G is the smallest positive integer k such that for each pair of vertices u and v there is a vertex w such that there exist a walk of length k connecting u and w and a walk of length k connecting v and w. For a primitive graph G with smallest cycle C_s of length s, we present an upper bound on the scrambling index of G that depends on s and the maximum distance between vertices in G and the cycle C_s. We then classify the graphs that satisfy the upper bound.

1. Introduction

Let $G(V, E)$ denote a simple graph on n vertices. We follow [1,2] for terminologies on graph. A walk W connecting u and v is denoted by W_{uv}. A walk connecting u and v is closed whenever $u = v$, and is open otherwise. A path P_{uv} connecting u and v is a walk W_{uv} with distinct vertices, except possibly $u = v$. A cycle is a closed path. The length of a walk W_{uv} by $\ell(W_{uv})$. A walk W_{uv} is also denoted by $u \overset{W_{uv}}{--} v$. For simplicity a walk of length k connecting u and v is denoted by $u \overset{k}{--} v$ walk. The distance between vertex u and vertex v in a connected graph, denoted $d(u, v)$, is the length of the shortest path connecting u and v. For any set $X \subseteq V(G)$ and a vertex $v \notin X$ the distance between v and X is defined by

$$d(v, X) = \min\{d(v, x) : x \in X\}.$$

If $v \in X$ we define $d(v, X) = 0$.

A connected graph G is primitive if there is a positive integer k such that for each pair of vertices u and v in G, there is a $u \overset{k}{--} v$ walk. The least of such positive integer k is the exponent of G and is denoted by $\exp(G)$. It is well known (see e.g. [1]) that a graph G is primitive if and only if G has a cycle of odd length.

In 2009, Akelbek and Kirkland [3,4] introduced the notion of scrambling index of graph. The *scrambling index* of a primitive graph G, denoted by $k(G)$, is the least positive integer k such that for any pair of distinct vertices u and v in G there exists a vertex w with the property that there is a $u \overset{k}{--} w$ walk and a $v \overset{k}{--} w$ walk. For a pair of distinct vertices u and v in G the local scrambling index of u and v is the number

$$k_{u,v}(G) = \min_{w \in V(G)} \{k : \text{ there are } u \overset{k}{--} w \text{ walk and } v \overset{k}{--} w \text{ walk}\}.$$

We note that if the local scrambling index of u and v is $k_{u,v}(G)$, then for any positive integer $\ell \geq k_{u,v}(G)$ we can find a vertex w' such that there is a walk $u \overset{\ell}{--} w'$ and $v \overset{\ell}{--} w'$. This implies

$$k(G) = \max_{u,v \in V(G)} \{k_{u,v}(G)\}.$$

Chen and Liu [5] have shown that for a primitive graph G on n vertices with the smallest cycle of length s, $k(G) \leq (s-1)/2 + n - s$. The purpose of this paper is to explore a different upper bound from Chen and Liu's bound, which in most case will be smaller than $(s-1)/2 + n - s$. In Section 2 we discuss some important properties of $u{-}{-}v$ walks. In Section 3 we present an upper bound for the scrambling index of primitive graphs. Finally, in Sections 4 we discuss classes of primitive graphs whose scrambling index achieved the upper bound.

2. Properties of walks

In this section we discuss some properties of $u{-}{-}v$ walk with end vertices u and v in some primitive graph.

Theorem 1. [2] *Let G be a graph. Every $u{-}{-}v$ walk in G contains a $u{-}{-}v$ path.*

Theorem 1 basically saying that we can shorten a $u{-}{-}v$ walk into a shorter $u - -v$ walk. The following result guarantees that we can lengthen a $u{-}{-}v$ walk into a longer $u{-}{-}v$ walk with the same parity.

Proposition 2. *Let G be a graph and u and v be two vertices in G. Every $u \overset{t}{--} v$ walk can be extended to a $u \overset{t+2m}{--} v$ walk for some positive integer m.*

Proof. Let u and v be vertices in G and let

$$W_{uv} : u = v_0 - v_1 - v_2 - \cdots - v_{t-1} - v_t = v$$

be a $u \overset{t}{--} v$ walk in G and let $W_2 : v - v_{t-1} - v$ be a closed walk of length 2. Then the walk

$$W'_{uv} : u \overset{W_{uv}}{--} v \overset{mW_2}{--} v$$

that starts a u, moves to v along the walk W_{uv}, and then moves m times around the closed walk $W_2 : v - v_{t-1} - v$ is a $u \overset{t+2m}{--} v$ walk. □

Proposition 3. *Let G be a graph. Then there is a $u \overset{2t}{--} v$ walk in G if and only if there is vertex w in G such that there are a $u \overset{t}{--} w$ walk and a $v \overset{t}{--} w$ walk in G.*

Proof. Suppose there is a vertex w in G such that there are a $u \overset{t}{--} w$ walk and a $v \overset{t}{--} w$ walk in G. Then there is a $w \overset{t}{--} v$ walk in G. This implies the walk $W_{uv} : u \overset{t}{--} w \overset{t}{--} v$ is a $u \overset{2t}{--} v$ walk.

Assume now that

$$W_{uv} : u = v_0 - v_1 - v_2 - \cdots - v_{2t-1} - v_{2t} = v$$

is a $u \overset{2t}{--} v$ walk in G. If we choose $w = v_t$, then there is a vertex w such that there are $u \overset{t}{--} w$ walk and a $v \overset{t}{--} w$ walk in G. □

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3. Bounds for scrambling index

We present a lower bound and an upper for scrambling index of primitive graphs.

Theorem 4. *Let G be a primitive graph and let C_s be a cycle of odd length s. If G does not have odd cyles with length smaller than s, Then*

$$k(G) \leq \frac{s-1}{2} + \max_{v \in V(G)} \{d(v, V(C_s))\}.$$

Proof. For each pair of distinct vertices u and v we show that there is a u $--$ v walk of length $(s-1) + 2\max_{v \in V(G)}\{d(v, V(C_s))\}$.

First, we claim that for any distinct vertices u and v in G, there is a path P_{uv} such that $\ell(P_{uv}) \leq (s-1)/2 + 2\max_{v \in V(G)}\{d(v, V(C_s))\}$. If both u and v lie on the cycle C_s, then there is a path P_{uv} with $\ell(P_{uv}) \leq (s-1)/2$. If v lies in $V(G) \setminus V(C_s)$ and u lies on C_s, then there is a path P_{uv} with $\ell(P_{uv}) \leq \max_{v \in V(G)}\{d(v, V(C_s))\}$. Suppose u and v lie on $V(G) \setminus v(C_s)$. Assume without loss of generality that $d(u, V(C_s)) \geq d(v, V(C_s))$ and that $d(u, V(C_s))$ is obtained by a path P_{uc} for some vertex c in C_s. If the vertex v lies on the path P_{uc}, then $\ell(P_{uv}) \leq \max_{v \in V(G)}\{d(v, V(C_s))\}$. Otherwise, there is a vertex y in C_s such that $d(v, y) = d(v, V(C_s))$. Since $d(u, c), d(v, y) \leq \max_{v \in v(G)}\{d(v, V(C_s))\}$, then the walk

$$W_{uv} : u \overset{P_{u,c}}{--} c \overset{P_{c,y}}{--} y \overset{P_{y,v}}{--} v$$

is a $u--v$ walk of length $\ell(W_{uv}) \leq (s-1)/2 + 2\max_{v \in V(G)}\{d(v, V(C_s))\}$. By Theorem 1 there is a path P_{uv} of length $\ell(P_{uv}) \leq (s-1)/2 + 2\max_{v \in V(G)}\{d(v, V(C_s))\}$.

We now show that the path P_{uv} can be extended to a $u--v$ walk of length exactly $(s-1) + 2\max_{v \in V(G)}\{d(v, V(C_s))\}$. If there is a $u--v$ path P_{uv} of even length, then Proposition 2 guarantees that we can extend the path P_{uv} to a $u--v$ walk of length exactly $(s-1) + 2\max_{v \in V(G)}\{d(v, V(C_s))\}$. We now assume that all $u--v$ paths are of odd lengths. Notice that u and v cannot be both on C_s. We show that we can extend a $u--v$ path into a $u--v$ walk of length exactly $(s-1) + 2\max_{v \in V(G)}\{d(v, V(C_s))\}$. We consider two cases.

Case 1: *There exists a path P_{uv} that has vertices in common with the cycle C_s*
We claim that the path P_{uv} and the cycle C_s have exactly one vertex in common. Suppose on the contrary that P_{uv} and C_s have more than one vertex in common. Let x_0 and y_0 be two vertices on C_s that lie on the path P_{uv}. We note that there are two paths on C_s say P_{x_0,y_0} and P'_{x_0,y_0} connecting u_0 and y_0. Since C_s is of odd length, $\ell(P_{x_0,y_0}) \not\equiv \ell(P'_{x_0,y_0})$ mod 2. This implies either the path $P_{uv} : u--x_0 \overset{P_{x_0,y_0}}{--} y_0--v$ or the path $P_{uv} : u--x_0 \overset{P'_{x_0,y_0}}{--} y_0--v$ is a $u--v$ path of even length. This contradicts the fact that all $u--v$ paths in G are of odd lengths.

We show that there is a $u--v$ walk W_{uv} of even length $\ell(W_{uv}) \leq (s-1) + 2\max_{v \in V(G)}\{d(v, V(C_s))\}$. Suppose the path P_{uv} and the cycle C_s have a vertex in common at v_0. If $v_0 = u$, then the walk $W_{uv} : u = v_0 \overset{C_s}{--} u = v_0 \overset{P_{u,v}}{--} v$ is a $u--v$ walk of even length $\ell(W_{uv}) \leq (s-1) + 2\max_{v \in V(G)}\{d(v, V(C_s))\}$. Similarly, if $v_0 = v$, then the walk $W_{uv} : u \overset{P_{u,v}}{--} v = v_0 \overset{C_s}{--} v = v_0$ is a $u--v$ walk of even length $\ell(W_{uv}) \leq (s-1) + 2\max_{v \in V(G)}\{d(v, V(C_s))\}$.

If $u \neq v_0$ and $v \neq v_0$, then the $u--v$ path P_{uv} can be decomposed into $u--v_0$ path P_{u,v_0} and v_0--v path $P_{v_0,v}$. Since all paths P_{uv} are of odd length, $\ell(P_{u,v_0}) + \ell(P_{v_0,v}) \leq 2\max_{v \in V(G)}\{d(v, V(C_s))\} - 1$. This implies the walk $W_{uv} : u \overset{P_{u,v_0}}{--} v_0 \overset{C_s}{--} v_0 \overset{P_{v_0,v}}{--} v$ is a $u--v$ walk of even length $\ell(W_{uv}) \leq (s-1) + 2\max_{v \in V(G)}\{d(v, V(C_s))\}$. Proposition 2 guarantees that we can extend the $u--v$ walk W_{uv} into a $u--v$ walk with length exactly

$(s-1)+2\max_{v\in V(G)}\{d(v,V(C_s))\}$.

Case 2: *All paths P_{uv} and the cycle C_s have no vertices in common*
There is a path P_{uv} and a vertex x_0 on P_{uv} and a vertex y_0 on C_s such that

$$d(x_0,y_0)=\min\{d(x,y): x \text{ on } P_{uv} \text{ and } y \text{ on } C_s\}.$$

Notice that the path P_{uv} can be decomposed into a path P_{u,x_0} connecting u and x_0, and a path $p_{x_0,v}$ connecting x_0 and v. Since $\ell(P_{uv})$ is odd, the walk

$$u \overset{P_{u,x_0}}{--} x_0 \overset{P_{x_0,y_0}}{--} y_0 \overset{C_s}{--} y_0 \overset{P_{y_0,x_0}}{--} x_0 \overset{P_{x_0,v}}{--} v$$

is a $u--v$ walk of even length. Notice also that $\ell(P_{u,x_0}) \not\equiv \ell(P_{x_0,v}) \bmod 2$. This implies $\ell(P_{u,y_0}) \not\equiv \ell(P_{y_0,v}) \bmod 2$. Since $\ell(P_{u,y_0}), \ell(P_{y_0,v}) \le \max_{v\in V(G)}\{d(v,V(C_s))\}$, we have $\ell(P_{u,y_0})+\ell(P_{y_0,v}) \le 2\max_{v\in V(G)}\{d(v,V(C_s))\}-1$. Thus $\ell(W_{uv}) \le s-1+2\max_{v\in V(G)}\{d(v,V(C_s))\}$. Proposition 2 guarantees that there is a $u--v$ walk W_{uv} with length equals $(s-1)+2\max_{v\in V(G)}\{d(v,V(C_s))\}$

Now Proposition 3 guarantees that for each pair of vertices u and v there is a vertex w such that there is a $u \overset{t}{--} w$ walk and there is a $v \overset{t}{--} w$ walk with $t=(s-1)/2+\max_{v\in V(G)}\{d(v,V(C_s))\}$. Thus the scrambling index $k(G) \le (s-1)/2+\max_{v\in V(G)}\{d(v,V(C_s))\}$. □

We note that for a primitive graph G with the smallest ood cycle of length s, $\max_{v\in V(G)}\{d(v,V(C_s))\} \le n-s$. Hence the bound given in Theorem 4 is smaller than or equal to the bound $(s-1)/2+(n-s)$.

4. Primitive graphs achieving the upper bound

In this section we discuss classes of primitive graphs that satisfy the upper bound given in Theorem 4. We first characterize and then discuss instances of such primitive graphs.

Corollary 5. *Let G be a primitive graph and let C_s be a smallest odd cycle of length s in G. The scrambling index*

$$k(G)=\frac{s-1}{2}+\max_{v\in V(G)}\{d(v,V(C_s))\}$$

if and only if there are vertices u_0 and v_0 such that the shortest even u_0--v_0 walk is a walk of length $(s-1)+2\max_{v\in V(G)}\{d(v,V(C_s))\}$.

Proof. Assume that there are two distinct vertices u_0 and v_0 in G such that the shortest even walk connecting u_0 and v_0 is of length $s-1+2\max_{v\in V(G)}\{d(v,V(C_s))\}$. Then by Proposition 3 there exists a vertex w such that there is a $u_0 \overset{t}{--} w$ walk and a $v_0 \overset{t}{--} w$ walk with $t=(s-1)/2+\max_{v\in V(G)}\{d(v,V(C_s))\}$. We claim that $k_{u_0,v_0}(G)=(s-1)/2+\max_{v\in V(G)}\{d(v,V(C_s))\}$. Suppose on the contrary that $k_{u_0,v_0}(G)=\ell$ for some positive integer $\ell<(s-1)/2+\max_{v\in V(G)}\{d(v,C_s)\}$. Then there exists a vertex w' with the property that there is a $u_0 \overset{\ell}{--} w'$ walk and a $v_0 \overset{\ell}{--} w'$ walk. But this implies the u_0--v_0 walk $W_{u_0,v_0}: u_0 \overset{\ell}{--} w' \overset{\ell}{--} v_0$ is a u_0--v_0 walk of even length $2\ell<(s-1)+2\max_{v\in V(G)}\{d(v,V(C_s))\}$. This contradicts the fact that the shortest even u_0--v_0 walk is of length $(s-1)+2\max_{v\in V(G)}\{d(v,V(C_s))\}$. Therefore,

$$k_{u_0,v_0}(G)=(s-1)/2+\max_{v\in V(G)}\{d(v,V(C_s))\}$$

and hence

$$k(G) \geq k_{u_0,v_0}(G) = (s-1)/2 + \max_{v \in V(G)}\{d(v, V(C_s))\}.$$

By Theorem 4 we conclude that $k(G) = (s-1)/2 + \max_{v\in V(G)}\{d(v, V(C_s))\}$.

We now assume that $k(G) = (s-1)/2+\max_{v\in V(G)}\{d(v, V(C_s))\}$. Then there are two distinct vertices u_0 and v_0 such that

$$k_{u_0,v_0}(G) = \min_{w\in V(G)}\{t : \text{there are } u_0 \overset{t}{--} w \text{ and } v_0 \overset{t}{--} w \text{ walks}\} = k(G).$$

This implies the walk $u_0 \overset{k(G)}{--} w \overset{k(G)}{--} v_0$ is the shortest walk of even length $(s-1) + 2\max_{v\in V(G)}\{d(v, V(C_s))\}$ connecting u_0 and v_0. □

We next discuss the class of primitive graph with smallest cycle of length s that satisfies the bound in Theorem 4. For that purpose we need the following definition. An open path P of length $\ell(P) = \max_{v\in V(G)}\{d(v, V(C_s))\}$ with one end vertex in C_s is called to be *special* if it does not have vertices in common with cycle of odd length other than C_s.

Corollary 6. *Let G be a primitive graph with the shortest odd cycle of length s. If G has a special path, then $k(G) = (s-1)/2 + \max_{v\in V(G)}\{d(v, V(C_s))\}$.*

Proof. Let P_{v_0,u_0} be a special path in G and let $u_0 \in V(C_s)$ and $v_0 \in V(G)$. Let v_0-y_0 be an edge of P_{v_0,u_0} such that $d(v_0, u_0) > d(y_0, u_0)$. Then the v_0--y_0 walk $W_{v_0,y_0} : v_0 \overset{P_{v_0,u_0}}{--} u_0 \overset{C_s}{--} u_0 \overset{P_{u_0,y_0}}{--} y_0$ is the shortest walk connecting v_0 and y_0 of even length $(s-1) + 2\max_{v\in V(G)}\{d(v, V(C_s))\}$. Corollary 5 implies that $k(G) = (s-1)/2 + \max_{v\in V(G)}\{d(v, V(C_s))\}$. □

Corollary 7. *Let G be a primitive graph containing a unique cycle of odd length. Let C_s be the odd cycle in G say of length s. Then $k(G) = (s-1)/2 + \max_{v\in V(G)}\{d(v, V(C_s))\}$.*

Proof. Since G has only one cycle of odd length, G must contain a special path. The conclusion follows from Corollary 6. □

Let G be a loopless primitive graphs on n vertices with the smallest cycle of length s. Since $s \geq 3$ and $\max_{v\in V(G)}\{d(v, V(C_s))\} \leq n-3$, then by Theorem 4 we have $k(G) \leq n-2$. Let SI_n denote the set of positive integers t for which there exists a loopless primitive graph on n vertices with scrambling index equals t.

Corollary 8. *For any positive integer $n \geq 3$, $SI_n = \{1, 2, \ldots, n-2\}$.*

Proof. For positive integer t, $3 \leq t \leq n-1$, we define a primitive graph G_t on n vertices $\{v_1, v_2, \ldots, v_n\}$ to be the graph with edge set

$$\begin{aligned} E(G_t) &= \{v_1 - v_2 - v_3 - v_1\} \cup \{v_3 - v_4 - \cdots - v_{t-1} - v_t\} \\ &\quad \cup \{v_t - v_{t+i} : i = 1, 2, \ldots, n-t\} \end{aligned}$$

as shown in Figure 1.

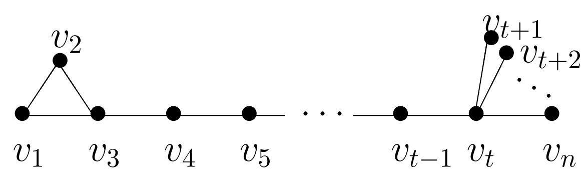

Figure 1. The graph G_t

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The graph G_t has a unique odd cycle of length 3. Moreover, G_t has special paths of length $t-2$. Corollary 6 implies that $k(G_t) = t-1$. Since $3 \leq t \leq n-1$ and $k(K_n) = 1$, then for each positive integer $1 \leq p \leq n-2$, there exists primitive graph $k(G) = p$. Hence $SI_n = \{1, 2, \ldots, n-2\}$. □

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